

Generalizations of Nekrasov-Okounkov Identity

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Abstract

Nekrasov-Okounkov identity gives a product representation of the sum over partitions of a certain function of partition hook length. In this paper we give several generalizations of the Nekrasov-Okounkov identity using the cyclic symmetry of the topological vertex.

1 Introduction

The Nekrasov-Okounkov identity was discovered by Nekrasov and Okounkov while studying the supersymmetric gauge theories [1]. It relates the powers of Euler products to the sum over product of partition hook lengths:

$$\sum_{\lambda} z^{|\lambda|} \prod_{(i,j) \in \lambda} \frac{h(i,j)^2 - t^2}{h(i,j)^2} = \prod_{k \geq 1} (1 - z^k)^{t^2 - 1} \quad (1.1)$$

where λ is the partition of $n \in \mathbb{Z}^+ \cup \{0\}$ and we denote by λ_i the parts of λ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$. $\ell(\lambda)$ denotes the number of non-zero parts of λ and $|\lambda| = \sum_i \lambda_i$ denotes the size of the partition. We can represent this partition as a Young diagram such that there are λ_i boxes in the i -th column. We denote by λ both the partition and its Young diagram. The box in a Young diagram has coordinates $(i, j) \in \lambda$ if $1 \leq i \leq \ell(\lambda)$, $1 \leq j \leq \lambda_i$. We denote the transpose of the partition λ by λ^t which is obtained by interchanging the columns and the rows of λ . The arm length $a(i, j)$, the leg length $\ell(i, j)$ and the hook length $h(i, j)$ are given by

$$\begin{aligned} a_{\lambda}(i, j) &= \lambda_j^t - i, \quad \ell_{\lambda}(i, j) = \lambda_i - j \\ h_{\lambda}(i, j) &= a(i, j) + \ell(i, j) + 1 = \lambda_i + \lambda_j^t - i - j + 1 \end{aligned} \quad (1.2)$$

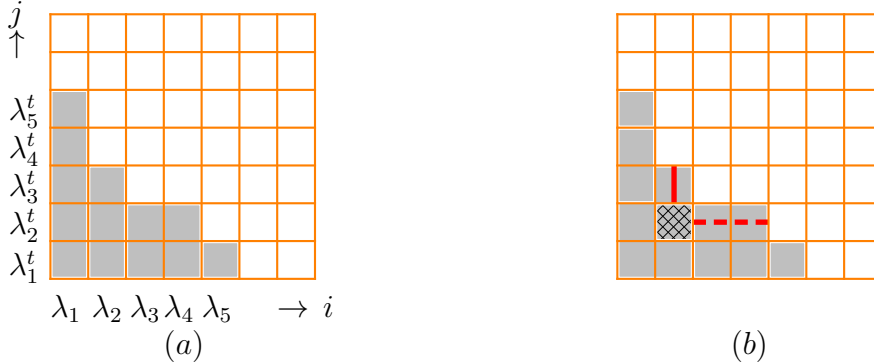


Figure 1: (a) Young diagram for $\lambda = \{5, 3, 2, 2, 1\}$. The transpose of this is $\lambda^t = \{5, 4, 2, 1, 1\}$. (b) The horizontal dotted line is the arm length and the vertical line is the leg length of the box with coordinates $(2, 2)$.

In an earlier paper [2] we stated a two-parameter generalization of NO-identity Eq(1.1):

$$\sum_{\lambda} x^{|\lambda|} \prod_{s \in \lambda} \frac{(a(s) + 1 + \vartheta \ell(s) - t)(a(s) + \vartheta(\ell(s) + 1) + t)}{(a(s) + 1 + \vartheta \ell(s))(a(s) + \vartheta(\ell(s) + 1))} = \prod_{k \geq 1} (1 - x^k)^{(t-1)(\frac{t}{\vartheta} + 1)} \quad (1.3)$$

This identity also comes from the study of the Seiberg-Witten theory and its relation with random partitions [2]. In this paper we will give other generalizations of Nekrasov-Okounkov identity based on the topological vertex formalism [3–5] for computing the partition function of the Seiberg-Witten theory. We will not discuss the physics behind these identities and will therefore not discuss the Seiberg-Witten theory but will rather use the topological vertex and its symmetries as a tool to generate such identities.

The topological vertex is a function of a complex parameter q , indexed by three partitions and defined in terms of the Schur and skew-Schur functions, see [5]¹ :

$$C_{\lambda\mu\nu}(q) = q^{\frac{\kappa(\mu)}{2}} s_{\nu^t}(q^{-\rho}) \sum_{\eta} s_{\lambda^t/\eta}(q^{-\rho-\nu}) s_{\mu/\eta}(q^{-\rho-\nu^t}) \quad (1.4)$$

where $s_{\nu/\eta}(x_1, x_2, \dots)$ is the skew-Schur function, $q^{-\rho-\nu} = \{q^{\frac{1}{2}-\nu_1}, q^{\frac{3}{2}-\nu_2}, q^{\frac{5}{2}-\nu_3}, \dots\}$ and $\kappa(\nu) = \sum_i \nu_i(\nu_i + 1 - i)$. Notice that λ and μ appear to be almost on the same footing but ν is treated very differently. An important property of the topological vertex, which is not obvious from the definition given above, is that it is cyclically symmetric:

$$C_{\lambda\mu\nu}(q) = C_{\mu\nu\lambda}(q) = C_{\nu\lambda\mu}(q) \quad (1.5)$$

This cyclic symmetry implies interesting identities involving Schur functions,

$$s_{\mu}(q^{-\rho}) s_{\lambda}(q^{-\rho-\mu}) = q^{-\frac{\kappa(\mu)+\kappa(\lambda)}{2}} \sum_{\eta} s_{\lambda^t/\eta}(q^{-\rho}) s_{\mu^t/\eta}(q^{-\rho}) \quad (1.6)$$

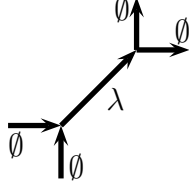
The topological vertex is a combinatorial object and can be interpreted as the generating function of plane partitions (3D partitions) with certain constraints. This combinatorial definition makes the cyclic symmetry manifest as discussed in [5].

2 Derivation of Nekrasov-Okounkov Identity

In this section we will derive the Nekrasov-Okounkov identity. The method of generalization this identity will also be discussed.

Consider the following oriented trivalent graph:

¹We will use the definition of the topological vertex given in [5] which differs slightly from the one given in [4].



We will associate with each edge of the graph a partition λ and a factor $z^{|\lambda|}$ such that the empty partition \emptyset is associated with the external edges. With each vertex we will associate the ordered triplet $\{\lambda, \mu, \nu\}$ and a factor $C_{\lambda\mu\nu}(q)$ where λ, μ and ν are the partitions associated with the edges of that vertex such that the edges are directed into the vertex. If an edge is outgoing from a vertex we replace the partition λ associated with that edge with λ^t . We take the convention that the order is given by anticlockwise going around the vertex. We will see later that since the topological vertex is cyclically symmetric where we begin in order to go around anticlockwise will not matter and this freedom will give rise to identities. A partition λ associated with an internal edge will be present in two different triplets and we take the convention that when a partition occurs a second time in a triplet it is replaced with its transpose (this can be understood by considering oriented graphs).

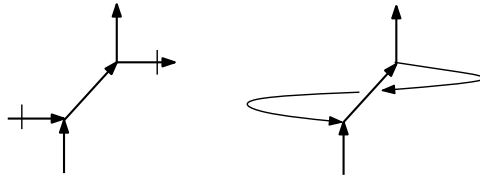
Thus for the above graph we get the factor:

$$z^{|\lambda|} C_{\emptyset\emptyset\lambda}(q) C_{\emptyset\emptyset\lambda^t}(q) = z^{|\lambda|} C_{\lambda\emptyset\emptyset}(q) C_{\lambda^t\emptyset\emptyset}(q) = z^{|\lambda|} C_{\emptyset\lambda\emptyset}(q) C_{\lambda^t\emptyset\emptyset}(q) \quad (2.1)$$

where the equality follows from the cyclic symmetry of the vertex. By summing over all non-trivial partition we get the function associated with the graph:

$$\begin{aligned} \mathcal{Z} &= \sum_{\lambda} z^{|\lambda|} C_{\emptyset\emptyset\lambda}(q) C_{\emptyset\emptyset\lambda^t}(q) \\ &= \sum_{\lambda} z^{|\lambda|} s_{\lambda^t}(q) s_{\lambda}(q) = \prod_{i,j=1}^{\infty} (1 + z q^{i+j-1}) = \prod_{k=1}^{\infty} (1 + z q^k)^k \end{aligned} \quad (2.2)$$

Now lets consider the graph obtained from Fig. 2 by gluing two horizontal external edges as shown in the figure below.



In this case the function associated with this modified graph is given by

$$\begin{aligned}\mathcal{Z} &= \sum_{\tau, \sigma} z_1^{|\sigma|} z_2^{|\tau|} C_{\emptyset \sigma \tau^t}(q) C_{\emptyset \sigma^t \tau}(q) \\ &= \sum_{\tau, \sigma} z_1^{|\sigma|} z_2^{|\tau|} s_{\tau}(q^{-\rho}) s_{\tau^t}(q^{-\rho}) s_{\sigma^t}(q^{-\rho-\tau^t}) s_{\sigma}(q^{-\rho-\tau})\end{aligned}\quad (2.3)$$

Using the summation identity for Schur function,

$$\sum_{\mu} q^{|\mu|} s_{\mu}(x) s_{\mu^t}(y) = \prod_{i,j \geq 1} (1 + q x_i y_j) \quad (2.4)$$

we can sum over σ to obtain

$$\mathcal{Z} = \sum_{\tau} z_2^{|\tau|} s_{\tau}(q^{-\rho}) s_{\tau^t}(q^{-\rho}) \prod_{i,j=1}^{\infty} \left(1 + z_1 q^{-\tau_i - \tau_j^t - \rho_i - \rho_j}\right) \quad (2.5)$$

The infinite product can be simplified using the identity

$$\begin{aligned}\frac{\prod_{i,j=1}^{\infty} \left(1 + z_1 q^{-\tau_i - \tau_j^t - \rho_i - \rho_j}\right)}{\prod_{i,j=1}^{\infty} \left(1 + z_1 q^{-\rho_i - \rho_j}\right)} &= \prod_{s \in \tau} \left((1 + z_1 q^{h(s)})(1 + z_1 q^{-h(s)})\right) \\ &= z_1^{|\tau|} q^{-\frac{\|\lambda\|^2 + \|\tau^t\|^2}{2}} \prod_{s \in \tau} \left((1 + z_1 q^{h(s)})(1 + z_1^{-1} q^{h(s)})\right)\end{aligned}\quad (2.6)$$

Using the above identity and the principal specialization of the Schur function,

$$s_{\lambda}(q^{-\rho}) = q^{\frac{\|\lambda\|^2}{2}} \prod_{s \in \lambda} (1 - q^{h(s)})^{-1}, \quad \|\lambda\|^2 = \sum_i \lambda_i^2 \quad (2.7)$$

in Eq(2.5) we get

$$\mathcal{Z} = \left(\prod_{k \geq 1} (1 + z_1 q^k)^k \right) \sum_{\tau} (z_1 z_2)^{|\tau|} \prod_{s \in \tau} \frac{(1 + z_1 q^{h(s)})(1 + z_1^{-1} q^{h(s)})}{(1 - q^{h(s)})^2} \quad (2.8)$$

Since $C_{\lambda \mu \nu}$ is cyclically symmetric

$$C_{\emptyset \sigma \tau^t}(q) = C_{\sigma \tau^t \emptyset}(q), \quad C_{\emptyset \sigma^t \tau}(q) = C_{\sigma^t \tau \emptyset}(q) \quad (2.9)$$

therefore we can write Eq(2.3) also as

$$\begin{aligned}\mathcal{Z} &= \sum_{\tau \sigma} z_1^{|\sigma|} z_2^{|\tau|} C_{\sigma \tau^t \emptyset}(q) C_{\sigma^t \tau \emptyset}(q) \\ &= \sum_{\sigma, \eta, \gamma, \tau} z_1^{|\sigma|} z_2^{|\tau|} s_{\sigma^t/\eta}(q^{-\rho}) s_{\sigma/\gamma}(q^{-\rho}) s_{\tau^t/\eta}(q^{-\rho}) s_{\tau/\gamma}(q^{-\rho})\end{aligned}\quad (2.10)$$

Using the summation identities for the skew-Schur function

$$\begin{aligned}
s_{\lambda/\mu}(x, y) &= \sum_{\nu} s_{\lambda/\nu}(x) s_{\nu/\mu}(y) \\
\sum_{\eta} s_{\eta^t/\nu}(x) s_{\eta/\mu}(y) &= \prod_{i,j \geq 1} (1 + x_i y_j) \sum_{\tau} s_{\mu^t/\tau}(x) s_{\nu^t/\tau^t}(y) \\
\sum_{\rho, \lambda} q^{|\rho|} s_{\rho/\lambda}(x) s_{\rho/\lambda}(y) &= \prod_{k \geq 1} (1 - q^k)^{-1} \prod_{i,j \geq 1} (1 - q^k x_i y_j)^{-1}
\end{aligned} \tag{2.11}$$

we get

$$\mathcal{Z} = \left(\prod_{k \geq 1} (1 + z_1 q^k)^k \right) \prod_{k \geq 1} (1 - z_1^k z_2^k)^{-1} \prod_{r \geq 1} \frac{(1 + z_1^{k-1} z_2^k q^r)^r (1 + z_1^{k+1} z_2^k q^r)^r}{(1 - z_1^k z_2^k q^r)^{2r}} \tag{2.12}$$

Comparing Eq(2.8) and Eq(2.12) we get the following identity

$$\sum_{\tau} (z_1 z_2)^{|\tau|} \prod_{s \in \tau} \frac{(1 + z_1 q^{h(s)})(1 + z_1^{-1} q^{h(s)})}{(1 - q^{h(s)})^2} = \prod_{k \geq 1} (1 - z_1^k z_2^k)^{-1} \prod_{r \geq 1} \frac{(1 + z_1^{k-1} z_2^k q^r)^r (1 + z_1^{k+1} z_2^k q^r)^r}{(1 - z_1^k z_2^k q^r)^{2r}}$$

The above is the generalized form of the Nekrasov-Okounkov identity. If we let

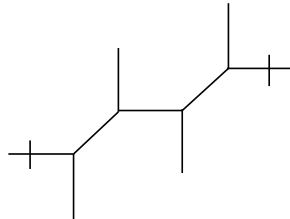
$$z_1 = -e^{\beta t}, \quad z_2 = -z, \quad q = e^{-\beta} \tag{2.13}$$

in the above identity then in the limit $\beta \mapsto 0$ we obtain the Nekrasov-Okounkov Identity:

$$\sum_{\tau} z^{|\tau|} \prod_{s \in \tau} \frac{h^2(s) - t^2}{h^2(s)} = \prod_{k \geq 1} (1 - z^k)^{t^2 - 1} \tag{2.14}$$

3 Nekrasov-Okounkov type identities

To illustrate the method of obtaining these identities we work out another example which leads to another Nekrasov-Okounkov type identity. The graph we will consider is given in the Fig. 3 below.



The function associated with this graph is given by

$$\begin{aligned}
\mathcal{Z} &= \sum_{\tau, \sigma, \nu, \lambda} z_1^{|\sigma|} z_2^{|\tau|} z_3^{|\lambda|} z_4^{|\nu|} C_{\sigma^t \tau \emptyset}(q) C_{\sigma \nu^t \emptyset}(q) C_{\lambda^t \nu \emptyset}(q) C_{\lambda \tau^t \emptyset}(q) \\
&= \sum_{\substack{\tau, \sigma, \nu, \lambda, \\ \alpha, \beta, \gamma, \eta}} z_1^{|\sigma|} z_2^{|\tau|} z_3^{|\lambda|} z_4^{|\nu|} s_{\sigma/\eta}(q^{-\rho}) s_{\tau/\eta}(q^{-\rho}) s_{\sigma^t/\gamma}(q^{-\rho}) \\
&\quad s_{\nu^t/\gamma}(q^{-\rho}) s_{\lambda/\beta}(q^{-\rho}) s_{\nu/\beta}(q^{-\rho}) s_{\lambda^t/\alpha}(q^{-\rho}) s_{\tau^t/\alpha}(q^{-\rho}) \\
&= \prod_{k \geq 1} (1 - z^k)^{-1} \prod_{r \geq 1} \left(\frac{\prod_{a=1}^4 (1 + z^{k-1} z_a q^r) (1 + z^k z_a^{-1} q^r)}{(1 - z^k q^r)^4 (1 - \frac{z^k}{z_1 z_2} q^r) (1 - \frac{z^k}{z_2 z_3} q^r) (1 - \frac{z^k}{z_3 z_4} q^r) (1 - \frac{z^k}{z_4 z_1} q^r)} \right)^r
\end{aligned} \tag{3.1}$$

where $z = z_1 z_2 z_3 z_4$. Using the cyclic symmetry of the topological vertex we can write the function associated with the graph Fig. 3 also as

$$\begin{aligned}
\mathcal{Z} &= \sum_{\tau, \sigma, \nu, \lambda} z_1^{|\sigma|} z_2^{|\tau|} z_3^{|\lambda|} z_4^{|\nu|} C_{\emptyset \sigma^t \tau}(q) C_{\emptyset \sigma \nu^t}(q) C_{\emptyset \lambda^t \nu}(q) C_{\emptyset \lambda \tau^t}(q) \\
&= \sum_{\tau, \sigma, \nu, \lambda} z_1^{|\sigma|} z_2^{|\tau|} z_3^{|\lambda|} z_4^{|\nu|} s_{\sigma}(q^{-\rho-\nu}) s_{\sigma^t}(q^{-\rho-\tau^t}) s_{\lambda^t}(q^{-\rho-\nu^t}) s_{\lambda}(q^{-\rho-\tau}) s_{\tau}(q^{-\rho}) s_{\tau^t}(q^{-\rho}) s_{\nu}(q^{-\rho}) s_{\nu^t}(q^{-\rho})
\end{aligned} \tag{3.2}$$

Summation over σ and λ can be carried out easily using Eq(2.4) and using the infinite product identity

$$\frac{\prod_{i,j=1}^{\infty} (1 + z_1 q^{-\nu_i - \tau_j^t + i + j - 1})}{\prod_{i,j=1}^{\infty} (1 + z_1 q^{i+j-1})} = \prod_{(i,j) \in \nu} (1 + z_1 q^{-\nu_j - \tau_i^t + i + j - 1}) \prod_{(i,j) \in \tau} (1 + z_1 q^{\nu_j^t + \tau_i - i - j + 1})$$

we get

$$\begin{aligned}
\mathcal{Z} &= \left(\prod_{r \geq 1} (1 + z_1 q^r)^r (1 + z_3 q^r)^r \right) \sum_{\tau, \nu} (z_1 z_4)^{|\tau|} (z_2 z_3)^{|\nu|} \\
&\quad \prod_{(i,j) \in \tau} \frac{(1 + z_1^{-1} q^{i+j-1-\nu_j^t - \tau_i})(1 + z_3 q^{i+j-1-\nu_j^t - \tau_i})}{(1 - q^{h(i,j)})^2} \prod_{(i,j) \in \nu} \frac{(1 + z_1 q^{i+j-1-\nu_j - \tau_i^t})(1 + z_3^{-1} q^{i+j-1-\nu_j - \tau_i^t})}{(1 - q^{h(i,j)})^2}
\end{aligned} \tag{3.3}$$

Comparing Eq(3.1) and Eq(3.3) we get the identity

$$\begin{aligned}
&\sum_{\tau, \nu} (z_1 z_4)^{|\tau|} (z_2 z_3)^{|\nu|} \prod_{(i,j) \in \tau} \frac{(1 + z_1^{-1} q^{i+j-1-\nu_j^t - \tau_i})(1 + z_3 q^{i+j-1-\nu_j^t - \tau_i})}{(1 - q^{h(i,j)})^2} \\
&\quad \prod_{(i,j) \in \nu} \frac{(1 + z_1 q^{i+j-1-\nu_j - \tau_i^t})(1 + z_3^{-1} q^{i+j-1-\nu_j - \tau_i^t})}{(1 - q^{h(i,j)})^2} \\
&= \prod_{k \geq 1} (1 - z^k)^{-1} \prod_{r \geq 1} \left(\frac{(1 + z_2 q^r)(1 + z_4 q^r) \prod_{a=1}^4 (1 + z^k z_a q^r) (1 + z^k z_a^{-1} q^r)}{(1 - z^k q^r)^4 (1 - \frac{z^k}{z_1 z_2} q^r) (1 - \frac{z^k}{z_2 z_3} q^r) (1 - \frac{z^k}{z_3 z_4} q^r) (1 - \frac{z^k}{z_4 z_1} q^r)} \right)^r
\end{aligned} \tag{3.4}$$

Let us define

$$z_1 = -e^{\beta t_1}, \quad z_3 = -e^{\beta t_3}, \quad z_2 = -x, \quad z_4 = -y, \quad q = e^{-\beta} \quad (3.5)$$

then in the limit $\beta \rightarrow 0$ the above identity reduces to

$$\begin{aligned} & \sum_{\tau, \nu} x^{|\nu|} y^{|\tau|} \prod_{s \in \tau} \frac{(\ell_\tau(s) + a_\nu(s) + 1 + t_3)(\ell_\tau(s) + a_\nu(s) + 1 - t_1)}{h^2(s)} \\ & \prod_{s \in \nu^t} \frac{(\ell_\nu(s) + a_\tau(s) + 1 + t_3)(\ell_\nu(s) + a_\tau(s) + 1 - t_1)}{h^2(s)} \\ & = \prod_{k \geq 1} (1 - x^k y^{k-1})^{t_1 t_3} (1 - x^{k-1} y^k)^{t_1 t_3} (1 - x^k y^k)^{t_1^2 + t_3^2 - 1} \end{aligned} \quad (3.6)$$

4 Conclusions

We have shown that Nekrasov-Okounkov identity and its generalizations can be obtained using the cyclic symmetry of the topological vertex. This method can be used to generate highly non-trivial identities involving Schur and Skew-Schur functions. These identities encode in them certain dualities of the underlying physical theories. It would be interesting to prove these identities using purely combinatorial methods as was carried out for the Nekrasov-Okounkov identity in [6]. This method also sheds light on a question of Richard Stanley. In [7] Stanley asked if the Nekrasov-Okounkov identity Eq(1.1) can be generalized such that left hand side is

$$\sum_{\lambda} z^{|\lambda|} \prod_{s \in \lambda} \prod_{a=1}^k \frac{h(s)^2 - t_a^2}{h(s)^2} \quad (4.1)$$

Using the method discussed before we can obtain the following identity:

$$\begin{aligned} & \sum_{\nu} (z x y)^{|\nu|} \prod_{s \in \nu} \frac{(1 + x q^{h(s)})(1 + x^{-1} q^{h(s)})(1 + y q^{h(s)})(1 + y^{-1} q^{h(s)})}{(1 - q^{h(s)})^4} \\ & = \sum_{\nu, \mu, \lambda, \eta_1, \eta_2, \eta_3, \eta_4} z^{|\nu|} x^{|\mu|} y^{|\lambda|} s_{\nu/\eta_1}(\mathbf{x}) s_{\nu^t/\eta_2}(\mathbf{x}) s_{\nu/\eta_3}(\mathbf{x}) s_{\nu^t/\eta_4}(\mathbf{x}) s_{\mu/\eta_2}(\mathbf{x}) s_{\mu^t/\eta_3}(\mathbf{x}) s_{\lambda/\eta_4}(\mathbf{x}) s_{\lambda^t/\eta_1}(\mathbf{x}) \end{aligned} \quad (4.2)$$

where $\mathbf{x} = q^{-\rho} = \{q^{1/2}, q^{3/2}, q^{5/2}, \dots\}$. The left hand side of this identity can indeed be reduced to Eq(4.1) (for $k = 2$) by taking a limit similar to the one in Eq(3.5) but the right hand side involves products of skew-Schur functions. We are not familiar with any identities which will allow such a product of skew-Schur function to be summed up in to a product. It would be interesting to find a product representation of this sum. The case $k > 2$ can be worked out as well and gives again similar sum over the product of skew-Schur functions.

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